# Factorization, Rényi entropies and the scaling of the entanglement spectrum: universal effects in one dimension

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EQuaM kick-off meeting, Vienna, November 4, 2013

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# Hamiltonian

Spin-1/2 Hamiltonian

$$H = \sum_i \left( J_x S_i^x S_{i+1}^x + J_y S_i^y S_{i+1}^y + J_z S_i^z S_{i+1}^z \right) - h \sum_i S_i^z$$

1. $S_i^\alpha$ ($\alpha = x, y, z$) it the spin operator actiong on the $i-th$ spin of the chain
2. Interplay between two-body interactions and external field
3. Invariance under spatial translations
4. Preserve of the $Z_2$ parity

# Factorized ground-state

Usually the ground-state of our Hamiltonian is a strongly entangled states There exist some choice of the Hamiltonian parameter for which it admits a fully separable ground state

$$|G_f\rangle = \bigotimes_{i=1}^{N} (\alpha_i|\uparrow\rangle + \beta_i|\downarrow\rangle)$$

- If $\alpha_i, \beta_i \neq 0 \quad \Rightarrow \quad$ Factorization of the ground state.
- If $\alpha_i = 0$ or $\beta_i = 0 \Rightarrow$ Saturation of the ground state.

# Relevance of the factorization

Is the factorized state relevant for our knowledge?

$$|G_f\rangle = \bigotimes_{i=1}^{N} (\alpha_i|\uparrow\rangle + \beta_i|\downarrow\rangle)$$

1. Factorization means existence of a symmetry broken phase.
2. Different separable states allow to characterize different phases
3. Entanglement driven quantum phase transition and entanglement driven order?

# Searching for a factorization point I

## Condition for the existence of a factorized eigenstate

For a system made by qubits, If a quantum state is fully factorized, than for each spins exist one unitary, traceless and Hermitian operators $U_i$ that left the state unchanged.

$$< G_f|U_i H U_i|G_f > - < G_f|H|G_f >= 0$$

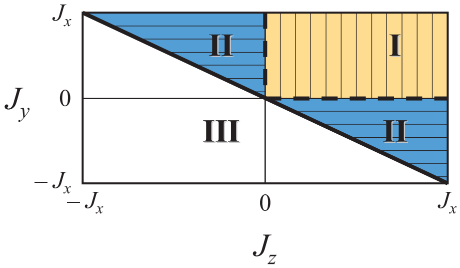
## Sufficient but non necessary conditions for a factorized eigenstate to be a ground-state

For any couple of spins, If the state obtained by the factorized state tracing out all the spins except two is the ground state of the local Hamiltonian, i.e the Hamiltonians made by the terms of that act only on the two spins, than the factorized state is the ground-state of the whole Hamiltonian.

# Searching for a factorization point II

$$H = \sum_i \left( J_x S_i^x S_{i+1}^x + J_y S_i^y S_{i+1}^y + J_z S_i^z S_{i+1}^z \right) - h \sum_i S_i^z$$

$$h_f = \sqrt{(J_x - J_z)(J_y - J_z)}$$



# Rényi entropies

Rényi entropies are measures of bipartite entanglement

$$S_\alpha(\ell) = \frac{1}{1-\alpha} \ln \left[ \mathrm{Tr} \left( \rho_\ell^\alpha \right) \right]$$
$$\rho_\ell = Tr_{N-\ell} |G\rangle\langle G|$$

- Play a relevant role in determining the scaling properties of numerical algorithms based on matrix product states
- are useful to determine the continuous or discontinuous nature of a phase
- the concept of topological entanglement entropy can be extended to the Rényi entropies

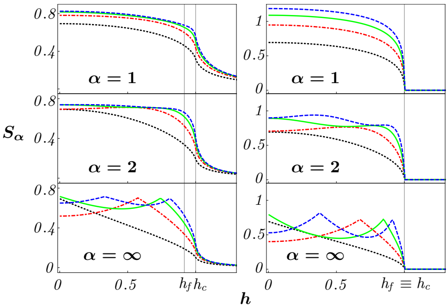
# XY model

$$H_{xy} = \frac{1}{2}\sum_i (1+\gamma)\sigma_i^x\sigma_{i+1}^x + (1-\gamma)\sigma_i^y\sigma_{i+1}^y - h\sum_i \sigma_i^z; \qquad \gamma = [0,1]$$

| | $\gamma = 0$ | $\gamma \neq 0$ |
|---|---|---|
| critical point | $h_c = 1$ | $h_c = 1$ |
| degeneracy of the gs below $h_c$ | no deg. | twofold deg. |
| order parameter | $\langle S_i^x \rangle = 0$ | $\langle S_i^x \rangle \neq 0$ |
| factorization | $h_f = 1$ | $h_f = \sqrt{1-\gamma^2}$ |
| Factorized ground state | 1 | 2 |

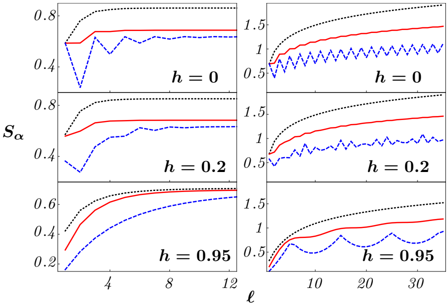
We will consider states of definite fixed parity

# Rényi entropies as function of $h$




- The von Neumann entropy $S_1(\ell, h)$, i.e. exhibits a monotonic behavior both in $h$ and in $\ell$.
- As soon as $\alpha > 1$ increase monotonicity in $h$ is removed for $h < h_f$
- For $\alpha > 2$ the scaling becomes non-monotonic also in $\ell$ for $h < h_f$: $S_\alpha(\ell+1, h) > S_\alpha(\ell, h)$.

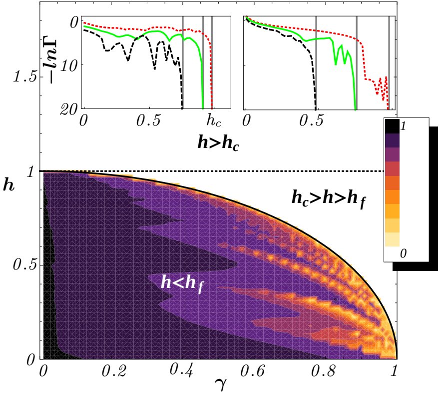
# Rényi entropies as function of $\ell$



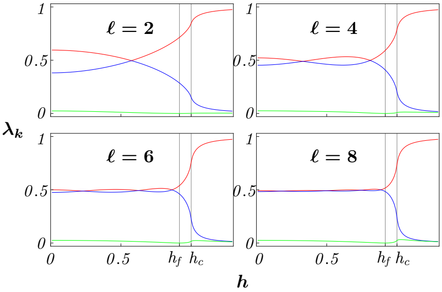

- For $h = 0$ the entropies of order $\alpha > 2$ exhibit damped oscillations that violate the area law behavior.
- As $h$ is increased, the frequency and the amplitude of the oscillations decrease; the area-law monotonic scaling is restored exactly at factorization.
- Oscillations are exponentially damped for $\gamma \neq 0$ noncritical models and algebraically damped in critical systems.

# Entanglement order parameter

$$\Gamma = -\sum_{\ell=1}^{\infty} \min[S_\infty(\ell+1,h) - S_\infty(\ell,h), 0] \, .$$

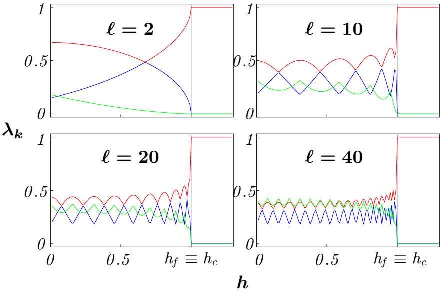


# Entanglement spectrum for non-critical models



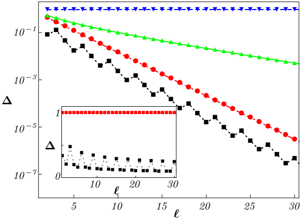

- The approach to quantum criticality is signaled by the opening of the Schmidt gap.
- For $h < h_f$ the two highest eigenvalues of $\rho_\ell$ spectrum exhibit an oscillating scaling similar to the Rényi entropies with $\alpha > 2$
- For partition of increasing $\ell$ the ending point of the oscillations converges, asymptotically, to $h_f$,

# Entanglement spectrum for critical models



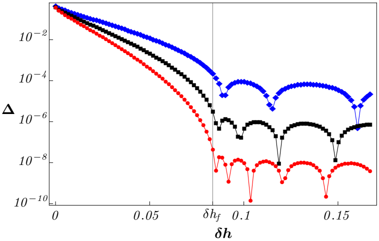

The behavior of the two highest eigenvalues is similar to the previous case but Differently from the previous case, the sum of the other eigenvalues cannot be neglected

# Schmidt Gap




- the Schmidt gap exhibits an overall envelope that is exponentially decreasing in the ordered phase of isotropic gapped models
- For $h_f < h < h_c$ the Schmidt gap undergoes a net exponential decay in $\ell$ which argument increase as $h$ decreases.
- for $h < h_f$ an oscillating behavior is superimposed on the exponential envelop while the argument of the exponential decay become independent from $h$

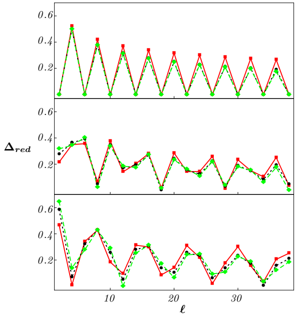
# Entanglement spectrum




The Schmidt gap as a function of $\delta h = |h - h_c|$ the frequency of the oscillations increases for growing $\ell$. The oscillatory scaling terminates at the value of $\delta h$ corresponding to the factorizing field, and $\Delta$ opens, independently of $\ell$, at the critical point $h_c$.

# Oscillation in the schmidt gap

$$\Delta_{red} = \Delta(h, \gamma)/\Delta(h_f, \gamma)$$




the oscillation frequency in the scaling of the reduced entanglement spectrum is a universal aspect common to critical and noncritical model

# Non exactly solvable models

$$H_{xyz} = \frac{1}{2}\sum_{i,l} J_x \sigma_i^x \sigma_l^x + J_y \sigma_i^y \sigma_l^y + J_z \sigma_i^z \sigma_l^z - h \sum_i \sigma_i^z$$

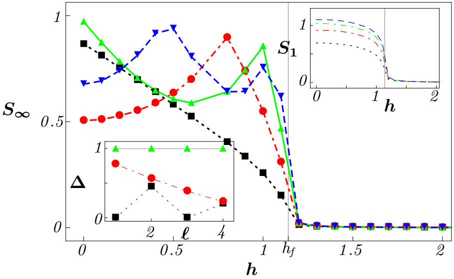
These systems undergo a quantum phase transition at $h = h_c$

factorized ground states do not necessarily exist. It exist only if

$$J_z \geq -J_y$$

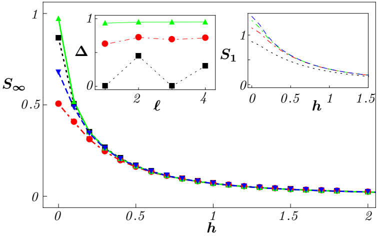
The system is not analytically solvable and to obtain information about ground state we need to diagonalize the system numerically

# Rényi entropies for systems with a factorization point




The qualitative behavior of these quantities is analogous to that of the *XY* models and confirms the existence of an entanglement-induced order of quasi-dimerized domains in the region $h < h_f$.

# Rényi entropies for systems without a factorization point




- the Rényi entropies behave smoothly and do not acquire local maxima
- The area-law scaling with the size of the block, $S_\alpha(\ell+1, h) > S_\alpha(\ell, h)$, appears to be *always* violated for *all* values of $h$
- as functions of the block size $\ell$ (at fixed external field $h$), the block entropies exhibit the same oscillating behavior as in models with factorized ground states

# Quantum frustration

- Many body systems are typically modeled by Hamiltonians that are sums of local terms.
- Each local term operates only on a part of the entire system and acts to minimize the corresponding energy.
- The competition among the different local terms can preclude the existence of configurations satisfying all minimizations simultaneously (Frustration)
- In quantum mechanics the non commutativity of different terms act as a new source of frustration

# Measure of frustration

$$f_S = 1 - \mathrm{tr}\left(\rho_G \cdot \Pi_S \otimes 1_R\right) = 1 - \mathrm{tr}_S\left(\rho_S \Pi_S\right)$$

- $\rho_S = \mathrm{tr}_R(\rho_G)$ is the reduced ground-state density matrix on $S$
- $\Pi_S$ is the projector onto the ground space of the local terms

For the Cauchy interlacing theorem

$$f_S \geqslant \epsilon_S^{(d)}, \qquad \epsilon_S^{(d)} = 1 - \sum_{i=1}^{d} \lambda_i^{\downarrow}(\rho_S)$$

$d$ is the degeneracy of the local ground space associated to subsystem $S$

# Physical interpretation of $\epsilon_S^{(d)}$ I

- For pure ground states $\epsilon_S^{(d)}$ is the distance of $\rho_G$ from the set of states with Schmidt rank less or equal than $d$.
- Such distance vanishes for all separable states and does not increase under local operations and classical communication (LOCC) (Entanglement Monotone)
- it becomes a faithful entanglement monotone only for $d = 1$ (bipartite geometric entanglement)
- Entanglement is a source of frustration

# Physical interpretation of $\epsilon_S^{(d)}$ II

In the presence of a mixed ground-state we have that $\epsilon_S^{(d)}$ is no more a measure of the entanglement.

we must replace by its convex roof

$$E_{S|R}^{(d)} = \inf_{\{p_k, |\psi_k\rangle\}} \sum_k p_k \epsilon_S^{(d)} \left(\mathrm{tr}_R |\psi_k\rangle\langle\psi_k|\right)$$

defining as a measure of classical correlations

$$C_{a|b}^{(d)} = \epsilon_a^{(d)} - \min_{\{M_b(x)\}} \sum_x p(x) \epsilon_a^{(d)} \left(\mathrm{tr}_b \rho(x)\right)$$

we have

$$\epsilon_S^{(d)} = E_{S|R}^{(d)} + C_{S|A}^{(d)}$$

Classical correlation is a source of frustration

# Frustration of a system I

$$f_s \geq \epsilon_S^{(d)} = E_{S|R}^{(d)} + C_{S|A}^{(d)}$$

Classification

- frustration free if $f_S = \epsilon_S^{(d)} = 0 \ \forall S$
- inequality saturated if $f_S = \epsilon_S^{(d)} \ \forall S$
- Inequality non-saturated if $\exists S | f_S > \epsilon_S^{(d)}$

# Frustration of a system II

Different groundstate of the same system may have different value of the frustration

Maximally mixed ground state

- convex combination with equal weights of all ground states
- proportional to the projector of ground space
- same symmetry of the Hamiltonian

- Frustration free in average if MMGS is frustration free
- Inequality saturated in average if MMGS is inequality saturated
- Inequality non-saturated in average if MMGS is inequality non-saturated

# Quantum Toulouse condition

**Prototype model**

$$H = \sum_{ij} h_{ij} = -\sum_{ij,\mu} J_{ij}^{\mu} S_i^{\mu} S_j^{\mu}$$

is a *prototype* model if

1. there exists at least one local ground space common to all pair local interactions $h_{ij}$
2. every local coupling vector $\vec{J}_{ij}$ has non-negative components.

Conjectures

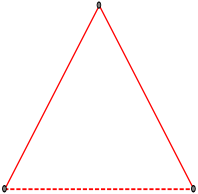
1. All prototype models are INES on average.
2. Every model obtained from a prototype model by local unitary operations on each spin and partial transposition on any arbitrary set of sites $\{K\}$ is still INES on average

# Partial Transposition

1. homogeneous nearest-neighbor ferromagnetic couplings
   - global gs is 5-fold degenerate, local gs is 3-fold degenerate
   - for any coupling $f_S = \epsilon_S^{(3)} = 0$
2. homogeneous nearest-neighbor anti-ferromagnetic couplings (PT on 2 and 4)
   - global and local gs are non-degenerate
   - for pairs $(1, 2)$ and $(3, 4)$ $f_S = \epsilon_S^{(1)} = 0.067$
   - for pair $(2, 3)$ $f_S = \epsilon_S^{(1)} = 0.5$

# Ines non-Ines transition I

$$H = \sum_{i,j} J_{i,j} S_i^x S_j^x + J_{i,j} S_i^y S_j^y$$



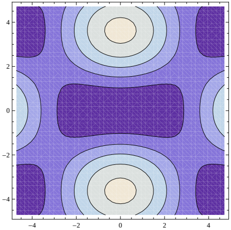
- $J_{1,2} = J_{2,3} = 1\ J_{1,3} = 0.99$
- $f_{1,2} = f_{2,3} = 0.25\ \epsilon^{(1)}_{1,2} = \epsilon^{(1)}_{2,3} = 0.25$
- $f_{1,3} = 1\ \epsilon^{(1)}_{1,3} = 0.665$

Non-Ines systems

- $J_{1,2} = J_{2,3} = 1\ J_{1,3} = 1.01$

# Ines non-Ines transition II

- Non-Ines System




- Ines System